Solutions

1. Given $\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, compute the following vectors and display them on an xy-plane: $\vec{u}$, $\vec{v}$, $\frac{1}{2}\vec{u}$, $-\vec{v}$, $\vec{u}+\vec{v}$, $\vec{u}-\vec{v}$, $\vec{u}-2\vec{v}$.

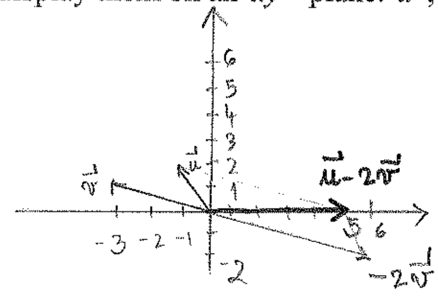
$$\frac{1}{2}\vec{u} = \frac{1}{2}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$$

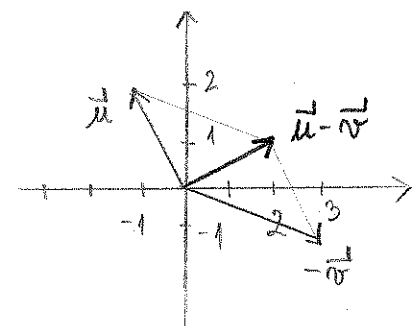
$$-\vec{v} = -\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

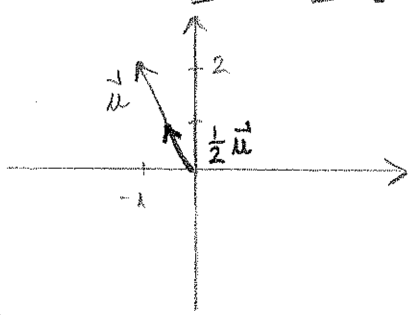
$$\vec{u}+\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

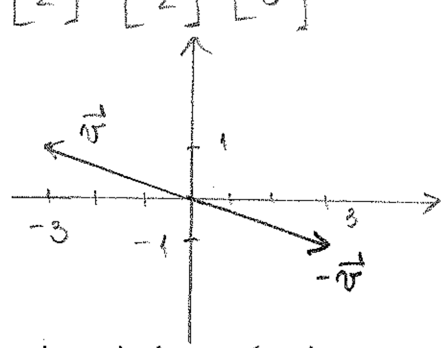
$$\vec{u}-\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

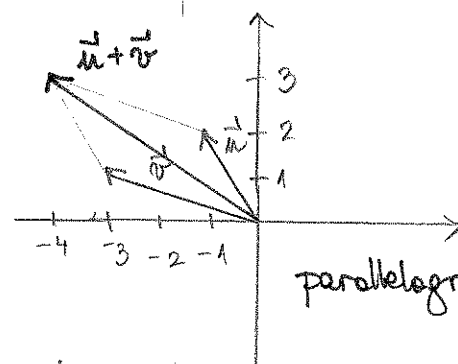
$$\vec{u}-2\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$











parallelogram law

2. Write a system of equations that is equivalent to the given vector equation:

i. $x_1\begin{bmatrix} 3 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} 7 \\ 3 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{cases} 3x_1 + 7x_2 - 2x_3 = 0 \\ -2x_1 + 3x_2 + x_3 = 0 \end{cases}$$

ii. $x_1\begin{bmatrix} 3 \\ -2 \\ 8 \end{bmatrix} + x_2\begin{bmatrix} 5 \\ 0 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$

$$\begin{cases} 3x_1 + 5x_2 = 2 \\ -2x_1 = -3 \\ 8x_1 - 9x_2 = 8 \end{cases}$$

3. Write a vector equation that is equivalent to the given system of equations:

i. $\begin{cases} x_2 + 5x_3 = 0 \\ 4x_1 + 6x_2 - x_3 = 0 \\ -x_1 + 3x_2 - 8x_3 = 0 \end{cases}$

$$x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

ii. $\begin{cases} 3x_1 - 2x_2 + 4x_3 = 3 \\ -2x_1 - 7x_2 + 5x_3 = 1 \\ 5x_1 + 4x_2 - 3x_3 = 2 \end{cases}$

$$x_1 \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -7 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

4. Determine if the vector $\vec{b}$ is a linear combination of the vectors $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$.

$$\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

$\vec{b}$ is a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$

$\Updownarrow$

equation $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$ has a sol.

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \Rightarrow \text{augmented matrix} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \sim$$

$$\overset{2r_1+r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right] \overset{-2r_2+r_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = 2 - 5x_3 \\ x_2 = 3 - 4x_3 \\ x_3 \text{ free} \end{cases}$$

$\Rightarrow$ System has solution, so $\vec{b}$ can be written as a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$. For example, when $x_3 = 0$, $x_1 = 2$, $x_2 = 3$ so $2\vec{a}_1 + 3\vec{a}_2 + 0\cdot\vec{a}_3 = \vec{b}$.
(∞-many choices for scalars x_1, x_2, x_3 are possible)

5. Let $\vec{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}$. For what values of h is $\vec{b}$ in the plane spanned by $\vec{a}_1$ & $\vec{a}_2$?

$\vec{b} \in \text{Span}\{\vec{a}_1, \vec{a}_2\} \Leftrightarrow \vec{b}$ can be written as a linear combination of $\vec{a}_1, \vec{a}_2 \Leftrightarrow \exists\, x_1, x_2 \in \mathbb{R}$ so that $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b} \Leftrightarrow [\vec{a}_1\ \vec{a}_2 \,\vdots\, \vec{b}]$ has a solution

$$\left[\begin{array}{cc|c} 1 & -5 & 3 \\ 3 & -8 & -5 \\ -1 & 2 & h \end{array}\right] \underset{r_1+r_3}{\overset{-3r_1+r_2}{\sim}} \left[\begin{array}{cc|c} 1 & -5 & 3 \\ 0 & 7 & -14 \\ 0 & -3 & 3+h \end{array}\right] \overset{\frac{1}{7}r_2}{\sim} \left[\begin{array}{cc|c} 1 & -5 & 3 \\ 0 & 1 & -2 \\ 0 & -3 & 3+h \end{array}\right] \overset{3r_2+r_3}{\sim} \left[\begin{array}{cc|c} 1 & -5 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & h-3 \end{array}\right]$$

if $[0\ 0 \,\vdots\, *]$ with $* \neq 0$ then system has no sol.

$\Rightarrow h - 3 = 0 \Rightarrow h = 3$

Sol. $\vec{b} \in \text{Span}\{\vec{a}_1, \vec{a}_2\}$ provided $h = 3$.

6. List five vectors in the $Span\{\vec{v}_1, \vec{v}_2\}$ and show the weights for $\vec{v}_1$ and $\vec{v}_2$ used to generate the vector:

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \ \vec{v}_1 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}.$$

$\vec{\omega} \in Span\{\vec{v}_1, \vec{v}_2\}$ iff $\exists\ x_1, x_2 \in \mathbb{R}$ s.t. $\vec{\omega} = x_1\vec{v}_1 + x_2\vec{v}_2$

$$\vec{\omega}_1 = 1\cdot\vec{v}_1 + 0\cdot\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{\omega}_2 = 0\cdot\vec{v}_1 + 1\cdot\vec{v}_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{\omega}_3 = 1\cdot\vec{v}_1 + 1\cdot\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

$$\vec{\omega}_4 = 3\vec{v}_1 - 2\vec{v}_2 = \begin{bmatrix} 9 \\ 3 \\ 6 \end{bmatrix} - \begin{bmatrix} -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 17 \\ 3 \\ 4 \end{bmatrix}$$

$$\vec{\omega}_5 = 5\vec{v}_1 + 0\cdot\vec{v}_2 = \begin{bmatrix} 15 \\ 5 \\ 10 \end{bmatrix}.$$

7. Let $\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} h \\ -5 \\ -4 \end{bmatrix}$. For what values of h is $\vec{b}$ in the plane spanned by $\vec{a}_1$ & $\vec{a}_2$?

for what h is

$$x_1 \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \cdot \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} h \\ -5 \\ -4 \end{bmatrix}$$

consistent?

$\Longleftrightarrow$

$$\left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -4 \end{array}\right] \text{ is consistent}$$

$$\sim \left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & -4+2h \end{array}\right] \begin{array}{c} \\ \\ r_3 + 2r_1 \end{array}$$

$$(0 \quad -2 \mid 10) \leftarrow -2r_2$$

$$\sim \left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 6+2h \end{array}\right] \begin{array}{c} \\ \\ r_3 - 2r_2 \end{array}$$

is consistent

$\Longleftrightarrow$

$6 + 2h = 0$

$\Longleftrightarrow$

$h = -3$

For each h above (which determines some $\vec{b}$ is in $Span\{\vec{a}_1, \vec{a}_2\}$), find the weights for $\vec{a}_1, \vec{a}_2$ used to generate this $\vec{b}$. Use this information to write a vector equation showing how to obtain $\vec{b}$ from $\vec{a}_1, \vec{a}_2$.

continue reducing

$$\sim \left[\begin{array}{cc|c} 1 & 0 & h-15 \\ 0 & 1 & -5 \\ 0 & 0 & 6+2h \end{array}\right] \begin{array}{c} r_1 + 3r_2 \\ \\ \\ \end{array}$$

$$\Longleftrightarrow \begin{cases} x_1 = h - 15 \\ x_2 = -5 \\ 0 = 6 + 2h \end{cases}$$

when $h = -3$

$x_1 = -18$

$x_2 = -5$

$0 = 0$

therefore

$$\underset{\uparrow\,\vec{a}_1}{-18 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}} + \underset{\uparrow\,\vec{a}_2}{(-5) \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}} = \underset{\uparrow\,\vec{b}}{\begin{bmatrix} -3 \\ -5 \\ -4 \end{bmatrix}}$$

Double check:

$$\begin{bmatrix} -18 \\ 0 \\ 36 \end{bmatrix} + \begin{bmatrix} 15 \\ -5 \\ -40 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ -4 \end{bmatrix} \checkmark$$

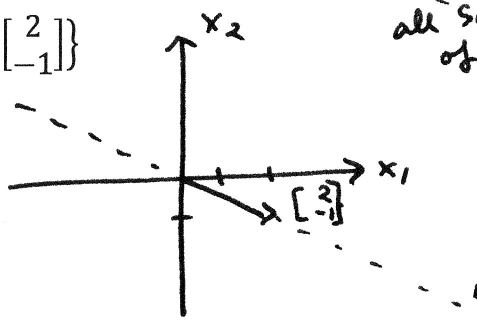
Remember: Span$\{\vec{v}_1\} = \{t\cdot\vec{v} : t \text{ is in } \mathbb{R}\}$ & Span$\{\vec{v}_1, \vec{v}_2\} = \{t\cdot\vec{v}_1 + s\cdot\vec{v}_2 : s,t \text{ in } \mathbb{R}\}$

all scalings of $\vec{v}_1$

all scalings & addings of $\vec{v}_1$ & $\vec{v}_2$

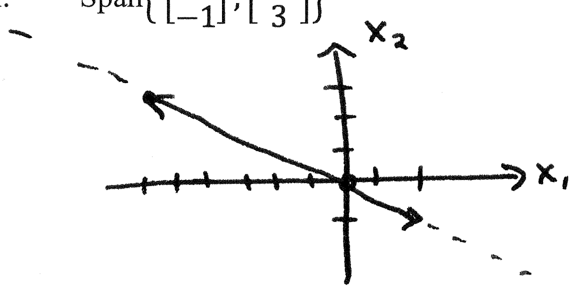
8. For each of the following, sketch *both* the generating vectors *and* their span on the same axes.

i. Span$\left\{\begin{bmatrix}2\\-1\end{bmatrix}\right\}$



span$\left\{\begin{bmatrix}2\\-1\end{bmatrix}\right\}$ is a line in $\mathbb{R}^2$ (2D)

ii. Span$\left\{\begin{bmatrix}2\\-1\end{bmatrix}, \begin{bmatrix}-6\\3\end{bmatrix}\right\}$

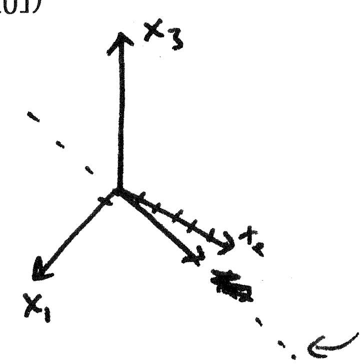


Notice: $\begin{bmatrix}-6\\3\end{bmatrix} = -3\cdot\begin{bmatrix}2\\1\end{bmatrix}$

$\Rightarrow$ their span is still a line

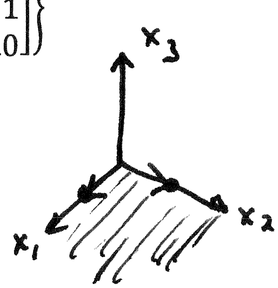
(same line as above)

iii. Span$\left\{\begin{bmatrix}1\\5\\0\end{bmatrix}\right\}$



span$\left\{\begin{bmatrix}1\\5\\0\end{bmatrix}\right\}$ is a line in $\mathbb{R}^3$ (3D)

iv. Span$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$



Span = whole x-y plane

9. Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in Span$\{\vec{u}, \vec{v}\}$ for all h and k.

Show $\left[\begin{array}{cc|c} 2 & 2 & h \\ -1 & 1 & k \end{array}\right]$ is consistent for all h,k

$\sim \left[\begin{array}{cc|c} -1 & 1 & k \\ 2 & 2 & h \end{array}\right]$

$\sim \left[\begin{array}{cc|c} -1 & 1 & k \\ 0 & 4 & h+2k \end{array}\right] \quad r_2 + r_1$

NOTE this cannot contain $[0 \; 0 \,|\, \blacksquare]$ regardless of h,k, because it does not even contain $[0 \; 0 \,|\, *]$

10. Mark each statement as ***True*** or ***False*** and justify your answer.

i. Another notation for the vector $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is $[-4 \quad 3]$

False

2x1 matrix

1x2 matrix

ii. The solution set of the linear system with augmented matrix $[\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \quad | \; \vec{b}]$ is the same as the solution of the system $\quad x_1 \cdot \vec{a}_1 + x_2 \cdot \vec{a}_2 + x_3 \cdot \vec{a}_3 = \vec{b}$

True!

iii. Asking whether the linear system corresponding to the linear system with augmented matrix $[\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \quad | \; \vec{b}]$ is consistent amounts to asking whether $\vec{b}$ is in Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

True. each are equivalent to asking if there are weights $x_1, \dots x_n$ s.t. $x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{b}$.

iv. Solving a system the linear system with augmented matrix $[\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \quad | \; \vec{b}]$ is the same as finding the weights (if any exist) that yield $\vec{b}$ as a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

True

v. If $\vec{u}, \vec{v} \neq \vec{0}$, then the set Span$\{\vec{u}, \vec{v}\}$ is always visualized as a plane through the origin.

False. If $\vec{u} = 2\vec{v}$, then Span$\{\vec{u}, \vec{v}\}$ is still a line.

7. Rewrite each of the following as a linear system, an augmented matrix, a vector equation and a matrix equation:

i. $\begin{cases} 4x_1 - x_2 = 8 \\ 5x_1 + 3x_2 = 2 \\ 3x_1 - x_2 = 1 \end{cases}$

linear system

$$\left[\begin{array}{cc|c} 4 & -1 & 8 \\ 5 & 3 & 2 \\ 3 & -1 & 1 \end{array}\right]$$

augmented matrix

$$x_1 \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}$$

matrix equation

$$\begin{bmatrix} 4 & -1 \\ 5 & 3 \\ 3 & -1 \end{bmatrix}_{3\times 2} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}_{3\times 1}$$

matrix equation

ii. $x_1 \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$

vector equation

$$\begin{cases} 4x_1 - 5x_2 + 7x_3 = 6 \\ -x_1 + 3x_2 - 8x_3 = -8 \\ 7x_1 - 5x_2 = 0 \\ -4x_1 + x_2 + 2x_3 = -7 \end{cases}$$

linear system

$$\left[\begin{array}{ccc|c} 4 & -5 & 7 & 6 \\ -1 & 3 & -8 & -8 \\ 7 & -5 & 0 & 0 \\ -4 & 1 & 2 & -7 \end{array}\right]$$

augmented matrix

$$\begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix}_{4\times 3} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}_{4\times 1}$$

matrix equation

iii. $\begin{bmatrix} 2 & -3 \\ 3 & 2 \\ 8 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}$ matrix equation

$\left[\begin{array}{cc|c} 2 & -3 & -21 \\ 3 & 2 & 1 \\ 8 & -5 & -49 \\ -2 & 1 & 11 \end{array}\right]$ augmented matrix

$\begin{cases} 2x_1 - 3x_2 = -21 \\ 3x_1 + 2x_2 = 1 \\ 8x_1 - 5x_2 = -49 \\ -2x_1 + x_2 = 11 \end{cases}$ linear system

$$x_1 \begin{bmatrix} 2 \\ 3 \\ 8 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}$$

vector equation

8. Note that $\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$ (*). Use this fact (and no row operations) to find scalars c_1, c_2 and c_3 so that $\begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix} = c_1\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$.

→ write vector form of equation (*): $-3\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + (-1)\begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$

$$\Rightarrow \begin{cases} c_1 = -3 \\ c_2 = -1 \\ c_3 = 2. \end{cases}$$

9. Determine if $\vec{b}$ is a linear combination of the columns of A:

$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$

$\vec{b}$ is a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$ $\Leftrightarrow$
$\Leftrightarrow$ $[\vec{a}_1\ \vec{a}_2\ \vec{a}_3 \mid \vec{b}]$ has a solution

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{array}\right] \overset{2r_1+r_2}{\sim} \left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

System has no sol.

$\vec{b}$ is NOT a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

10. Compute the following products; if a product is undefined explain why.

i. $\underset{3\times 2}{\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}} =$ undefined

ii. $\underset{3\times 1}{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 5 \\ -1 \end{bmatrix}} =$ undefined

iii. $\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 6 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} -2 \\ 3 \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix}}$

$1\cdot(-2) + 2\cdot 3 = -2+6 = 4$
$(-3)(-2) + 1\cdot 3 = 6+3 = 9$
$1\cdot(-2) + 6\cdot 3 = -2+18 = 16$

iv. $\underset{2\times 3}{\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 3 \\ 8 \end{bmatrix}}$

$1\cdot 1 + 3\cdot 2 - 4\cdot 1 = 1+6-4 = 3$
$3\cdot 1 + 2\cdot 2 + 1\cdot 1 = 3+4+1 = 8$

11. Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$. Does $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span R^3? Why or why not?

Theorem 4 / p. 37 $\quad \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3 \iff A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ has a pivot position in every row.

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix} \overset{r_1 \leftrightarrow r_3}{\sim} \begin{bmatrix} -3 & 9 & -6 \\ 0 & -3 & -2 \\ 0 & 0 & 4 \end{bmatrix} \overset{\substack{-\frac{1}{3}r_1 \\ -\frac{1}{3}r_2 \\ \frac{1}{4}r_3}}{\sim} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ has a pivot position in every row
$\underset{\text{Th. 4}}{\Rightarrow}$ $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$.

12. Can any vector in R^4 be written as a linear combination of the columns of the matrix $\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix}$?

equivalently, if $B = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4]$ does $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \mathbb{R}^4$?

$$B = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix} \overset{-2r_1+r_2}{\sim} \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 0 & 1 & 3 & -11 \end{bmatrix} \overset{\substack{-2r_2+r_3 \\ -r_2+r_4}}{\sim} \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & -7 \end{bmatrix} \sim$$

$$\overset{\frac{1}{15}r_3}{\sim} \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -7 \end{bmatrix} \overset{7r_3+r_4}{\sim} \begin{bmatrix} 1 & 4 & 4 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix $B = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4]$ does not have a pivot position in every row
$\underset{\text{Th. 4}}{\Rightarrow}$ not every vector in $\mathbb{R}^4$ is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$
$\Rightarrow$ Span ...

17. Let $\vec{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is $\vec{u}$ in the plane of $\mathbb{R}^3$ spanned by the columns of A? Why or why not?

Is $A\vec{x} = \vec{u}$ consistent?

i.e. is there x_1 & x_2 s.t. $x_1 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} -5 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$

Is $\left[\begin{array}{cc|c} 3 & 5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{array}\right]$ consistent?

$r_1 \leftrightarrow r_3$ $\sim \left[\begin{array}{cc|c} 1 & 1 & 4 \\ -2 & 6 & 4 \\ 3 & 5 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & 2 & -12 \end{array}\right] \begin{array}{l} \\ r_2 + 2r_1 \\ r_3 - 3r_1 \end{array}$

$(0 \quad -2 \quad -3) \leftarrow -\frac{1}{4} r_2$

$\sim \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & 0 & -15 \end{array}\right] r_3 - \frac{1}{4} r_2$

the matrix is inconsistent by theorem 2

$\Rightarrow$ NO weights x_1 & x_2 exist that get you $\vec{b}$

$\boxed{\Rightarrow \vec{b} \text{ is NOT in Span of columns of A}}$

18. Let $\vec{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{bmatrix}$. Is $\vec{u}$ in the plane of $\mathbb{R}^3$ spanned by the columns of A? Why or why not?

Is there x_1, x_2, x_3 so that

$x_1 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$

$\Leftrightarrow$ is $\left[\begin{array}{ccc|c} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{array}\right]$ consistent?

$r_1 \leftrightarrow r_3$ $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 8 & 7 & 2 \end{array}\right]$

$(-5 \quad -15 \quad 0 \quad -10) \leftarrow -5r_1$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -12 \end{array}\right] r_3 + (-5)r_1$ $\sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -33 \end{array}\right] r_3 + 7r_1$

$\vec{u}$ is NOT in column span of A

$\Uparrow$

NO weights exist that get $\vec{u}$ from A

$\Uparrow$

the system has no sln

-21
-12

19. Let $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Show that the equation $A\vec{x} = \vec{b}$ does not have a solution for all possible $\vec{b}$, and describe the set of all $\vec{b}$ for which $A\vec{x} = \vec{b}$ *does* have a solution.

$$\left[\begin{array}{cc|c} 2 & -1 & b_1 \\ -6 & 3 & b_2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 0 & -1 & b_1 \\ 0 & 0 & b_2+3b_1 \end{array}\right] \quad r_2+3r_1$$

the system is consistent

$\Leftrightarrow$

$b_2+3b_1=0$

$\Leftrightarrow$

$b_2=-3b_1$

---

the system has a solution $\Leftrightarrow \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ -3b_1 \end{bmatrix} = b_1 \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

for any real #.

20. Mark each statement as ***True*** or ***False*** and justify your answer.

i. The equation $A\vec{x} = \vec{b}$ is referred to as a *vector equation.*

False. it is a Matrix equation

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ii. A vector $\vec{b}$ is a linear combination of the columns of a matrix A if and only if the equation $A\vec{x} = \vec{b}$ has at least one solution.

True. By definition, $A\vec{x}=\vec{b}$ means $x_1\vec{a_1}+\cdots+x_n\vec{a_n}=\vec{b}$ where $A=[\vec{a_1}\cdots\vec{a_n}]$

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iii. If the columns of an $m \times n$ matrix A span $\mathbb{R}^m$ then the equation $A\vec{x} = \vec{b}$ is consistent for each $\vec{b}$ in $\mathbb{R}^m$

True. $A\vec{x}=\vec{b}$ has a solution

$\Leftrightarrow \vec{b}$ is a linear combo of columns of A

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iv. If A is an $m \times n$ matrix, and if the equation $A\vec{x} = \vec{b}$ is inconsistent for some $\vec{b}$ in $\mathbb{R}^m$ then A cannot have a pivot position in every row.

True. By theorem 4, Span$\{\vec{a_1}\ldots\vec{a_n}\}$ is all of $\mathbb{R}^m$

$\Leftrightarrow$ A has a pivot in every row.

---

v. If the equation $A\vec{x} = \vec{b}$ is inconsistent, then $\vec{b}$ is not in the set spanned by the columns of A.

these both mean

"$\vec{b}$ is not a linear combo of columns of A"